APPENDIX E

The Holdren Apportioning Method

A method for quantifying the respective contributions of population growth and changes in consumption per capita of any type of resource consumption was laid out in a landmark 1991 paper by Harvard physicist Prof. John Holdren.[21] Although Dr. Holdren's paper dealt specifically with the role of population growth in rising energy consumption, the method can be applied to many types of population/ resource consumption analyses. In the case of sprawl, the resource under consideration is rural land, namely the expansion over time of the Urbanized Area into rural areas.

As stated in Appendix D, the total land area occupied by the built-up Urbanized Area can be expressed as:

$$A = P \times a \qquad (1)$$

Where*:*

A = Area of total urbanized land in a city and its suburbs
a = area of urbanized land used by the average resident (per capita land use)
P = Population of that city and its suburbs

Following the logic in Holdren's paper, if over a period of time Δt (e.g., a year or decade), the population grows by an increment ΔP and the per capita land use changes by Δa, the total urbanized land area grows by ΔA which is given by substituting in eqn. (1):

$$A + \Delta A = (P + \Delta P) \times (a + \Delta a) \qquad (2)$$

Subtracting eqn. (1) from eqn. (2) and dividing through by A to compute the relative change (i.e., $\Delta A/A$) in urbanized land area over time interval Δt yields:

$$\Delta A/A = \Delta P/P + \Delta a/a + (\Delta P/P) \times (\Delta a/a) \qquad (3)$$

Now eqn. (3) is quite general and makes no assumption about the growth model or time interval. On a year-to-year basis, the percentage increments in P and a are small (i.e., single digit percentages), so the second order term in eqn. (3) can be ignored. Hence following the Holdren paradigm, eqn. (3) states that the percentage growth in urbanized land area (viz., 100% x $\Delta A/A$) is the sum of the percentage growth in the population (100% x $\Delta P/P$) plus the percentage growth in the per capita land use (100% x $\Delta a/a$). Stated in words, eqn. (3) becomes:

Overall percentage land area growth =
Overall percentage population growth + Overall percentage per capita growth (4)

In essence, the Holdren methodology quantifies population growth's share of total land consumption (sprawl) by finding the ratio of the overall percentage change in population over a period of time to the overall percentage change in land area consumed for the same period. This can be expressed as:

$$\text{Population share of growth} = \frac{(\textit{Overall percentage population growth})}{(\textit{Overall percentage land area growth})} \qquad (5)$$

The same form applies for per capita land use:

$$\text{Per cap. land use share of growth} = \frac{(\textit{Overall percentage per capita land use growth})}{(\textit{Overall percentage land area growth})} \qquad (6)$$

The above two equations follow the relationship based on Prof. Holdren's eqn. (5) in his 1991 paper. A common growth model follows the form (say for population):

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[21] See note 15.

$$P(t) = P_0 \; (1 + g_P)^t \qquad (7)$$

Where $P(t)$ is population at time t, P_0 is the initial population and g_P the growth rate over the interval. Solving for g_P the growth rate yields:

$$\ln(1 + g_P) = (1/t)\ln(P(t)/P_0) \qquad (8)$$

Since ln (1 + x) approximately equals x for small values of x, eqn. (8) can be written as:

$$g_P = (1/t)\ln(P(t)/P_0). \qquad (9)$$

The same form of derivation of growth rates can be written for land area (A) and per capita land use (a)

$$g_A = (1/t)\ln(A(t)/A_0) \qquad (10)$$

$$g_a = (1/t)\ln(a(t)/a_0). \qquad (11)$$

These three equations for the growth rates allow you to restate the Holdren result of eqn. (4) as:

$$g_P + g_a = g_A \qquad (12)$$

Substituting the formulae (eqns. 9 through 11) for the growth rates and relating the initial and final values of the variables P, a and A over the period of interest into eqn. (12), the actual calculational relationship becomes:

ln (*final population / initial population*) +

ln (*final per capita land area / initial per capita land area*) =

ln (*final total land area / initial total land area*) (13)

In other words, the natural logarithm (ln) of the ratio of the final to initial population, plus the logarithm of the ratio of the final to initial per capita land area (i.e., land consumption per resident), equals the logarithm of the final to the initial total land area.

In the case of the San Francisco-Oakland Urbanized Area from 1970 to 1990, this formula would appear as:

ln (*3,629,516 residents / 2,987,850 residents*) +

ln (*0.15413 acre per resident / 0.14587 acre per resident*) =

ln (*874.1 square miles / 681.0 square miles*) (14)

Computing the ratios yields:

$$\ln(1.215) + \ln(1.057) = \ln(1.284)$$

$$0.1950 + 0.0555 = 0.2505 \qquad (15)$$

Then applying eqns. (5) and (6), the percentage contributions of population growth and per capita land area growth are obtained by dividing (i.e., normalizing to 100%) each side by 0.2505:

$$\frac{0.1950}{0.2505} + \frac{0.0555}{0.2505} = \frac{0.2500}{0.2505} \qquad (16)$$

Performing these divisions yields:

$$0.78 + 0.22 = 1.0 \qquad (17)$$

Thus, we note that in the case of the San Francisco-Oakland Urbanized Area from 1970 to 1990, the share of sprawl due to population growth was 78% [100 % x (0.1950 / 0.250)], while declining density (i.e., an increase in land area per capita) accounted for 22% [100% x (0.0555 / 0.250)]. Note that the sum of both percentages equals 100%.

In a number of cases (28 out of the 100), the results of the Holdren method showed that either population growth or growth in per capita land consumption actually explained more than 100% of the sprawl that occurred, while the per capita land area growth (in the case of the former), or population growth (in the case of the latter) share was less than 0% (i.e., a negative number due either to higher population densities or a decline in population throughout the aggregate Urbanized Area). There were 17 cases in which population growth explained more than 100% of sprawl, and 11 cases in which growth in per capita land consumption did the same. Still in these instances, the sum of the percentage numbers – one positive and one negative – adds up to 100%. These are the cases in which overall population density increased, or alternatively, there was an absolute decline in population, throughout a given Urbanized Area. In Table 5 and Appendix A, to avoid confusion created by negative growth rates, the authors limited the calculated share of the total growth rate to no more than 100% and no less than 0% of sprawl. The issue is the percentage of a fixed number of square miles of sprawl that can be explained by one of the two factors. In layman's terms, 100% of those fixed square miles is the highest possible number.